5.1 
Inverse Functions

- Determine whether a function is one-to-one, and if it is, find a formula for its inverse.
- Simplify expressions of the type \((f \circ f^{-1})(x)\) and \((f^{-1} \circ f)(x)\).

Inverses

When we go from an output of a function back to its input or inputs, we get an inverse relation. When that relation is a function, we have an inverse function.

Interchanging the first and second coordinates of each ordered pair in a relation produces the inverse relation.

Consider the relation \(h\) given as follows:
\[ h = \{(-8, 5), (4, -2), (-7, 1), (3.8, 6.2)\}. \]

The inverse of the relation \(h\) is given as follows:
\[ \{(5, -8), (-2, 4), (1, -7), (6.2, 3.8)\}. \]
Ex. Consider the relation \( g \) given by
\[
g = \{(2, 4), (-1, 3), (-2, 0)\}.
\]
Find the inverse.

Inverse Relation

If a relation is defined by an equation, interchanging the variables produces an equation of the inverse relation.

Example

Find an equation for the inverse of the relation:
\[
y = x^2 - 2x.
\]

Equation: _________________

Practice:

Find an equation for the inverse of the relation: \( y = 3x - 8 \)

Equation: _________________
**Ex.**

Graphs of a Relation and Its Inverse

If a relation is given by an equation, then the solutions of the inverse can be found from those of the original equation by interchanging the first and second coordinates of each ordered pair. Thus the graphs of a relation and its inverse are always reflections of each other across the line \( y = x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = x^2 - 5x )</th>
<th>( x = y^2 - 5y )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>−1</td>
<td>6</td>
<td>6</td>
<td>−1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>−6</td>
<td>−6</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>−4</td>
<td>−4</td>
<td>4</td>
</tr>
</tbody>
</table>

**Practice:**

Graph the equation by substituting and plotting points. Then reflect the graph across the line \( y = x \) to obtain the graph of its inverse.

1. \( y = 3x - 2 \)
2. \( x = -y + 4 \)
3. \( y = |x| \)
One-to-One Functions

A function $f$ is **one-to-one** if different inputs have different outputs – that is,

$$\text{if } a \neq b, \text{ then } f(a) \neq f(b).$$

Or a function $f$ is **one-to-one** if when the outputs are the same, the inputs are the same – that is,

$$\text{if } f(a) = f(b), \text{ then } a = b.$$

**Ex.**

Given the function $f$, prove that $f$ is one-to-one using the definition of a one-to-one function.

1. $f(x) = \frac{1}{3}x - 6$
2. $f(x) = x^3 + \frac{1}{2}$
Horizontal-Line Test

If it is possible for a horizontal line to intersect the graph of a function more than once, then the function is not one-to-one and its inverse is not a function.

\[ f(a) = f(b), \quad a \neq b \]

not a one-to-one function

inverse is not a function

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Example

From the graph shown, determine whether each function is one-to-one and thus has an inverse that is a function.

a) \[ f(x) = 4 - x \]

No horizontal line intersects more than once: is one-to-one; inverse is a function

b) \[ f(x) = x^2 \]

Horizontal lines intersect more than once: not one-to-one; inverse is not a function
Example

From the graph shown, determine whether each function is one-to-one and thus has an inverse that is a function.

c) \[ f(x) = \sqrt{x + 2} + 3 \]

No horizontal line intersects more than once: is one-to-one; inverse is a function

d) \[ f(x) = 3x^5 - 20x^3 \]

Horizontal lines intersect more than once: not one-to-one; inverse is not a function

Ex.

Using the horizontal-line test, determine whether the function is one-to-one.

1.

2.
Ex + Practice:

Graph the function and determine whether the function is one-to-one using the horizontal-line test.

1. \( f(x) = 5x - 8 \)
2. \( f(x) = 1 - x^2 \)
3. \( f(x) = |x + 2| \)
4. \( f(x) = -\frac{4}{x} \)
5. \( f(x) = \frac{2}{3} \)
6. \( f(x) = \sqrt{25 - x^2} \)

Inverses of Functions

If the inverse of a function \( f \) is also a function, it is named \( f^{-1} \) and read “f-inverse.”

The \(-1\) in \( f^{-1} \) is not an exponent.

\( f^{-1} \) does not mean the reciprocal of \( f \) and \( f^{-1}(x) \) can not be equal to \( \frac{1}{f(x)} \).
One-to-One Functions and Inverses

- If a function \( f \) is one-to-one, then its inverse \( f^{-1} \) is a function.
- The domain of a one-to-one function \( f \) is the range of the inverse \( f^{-1} \).
- The range of a one-to-one function \( f \) is the domain of the inverse \( f^{-1} \).
- A function that is increasing over its domain or is decreasing over its domain is a one-to-one function.

Obtaining a Formula for an Inverse

If a function \( f \) is one-to-one, a formula for its inverse can generally be found as follows:
1. Replace \( f(x) \) with \( y \).
2. Interchange \( x \) and \( y \).
3. Solve for \( y \).
4. Replace \( y \) with \( f^{-1}(x) \).
Each graph is the graph of a one-to-one function $f$. Sketch the graph of the inverse function $f^{-1}$.

- **For each function:**

  a) Determine whether it is one-to-one.
  b) If the function is one-to-one, find a formula for the inverse.

  1. $f(x) = x + 4$
  2. $f(x) = x \sqrt{4 - x^2}$
  3. $f(x) = \frac{4}{x + 7}$
  4. $f(x) = 5x^2 - 2, \quad x \geq 0$